variation which caused it. Two points appear at once on examination of the theoretical curves of figs. 2 and 3:—

(*a*) The galvanometer deflection does not give a perfectly faithful, but a slightly distorted, reproduction of the E.M.F. curve.

(*b*) The galvanometer deflection lags behind the E.M.F. by an amount here equal to about one-fourth of the period.

Both these deviations from a true representation of the E.M.F. variation are due to the effect of the inertia of the moving system of the galvanometer.

On some New Formulæ for the Numerical Calculation of the Mutual Induction of Coaxial Circles.

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Section 1.—*Note on the Direct Numerical Calculation of Elliptic Integrals.*

The importance of realising rapid and accurate methods of calculating the elliptic integrals, now denoted by

$$\mathrm{F}(\phi, k) = \int_0^\phi d\phi/\Delta(\phi, k), \qquad \mathrm{E}(\phi, k) = \int_0^\phi \Delta(\phi, k)\, d\phi, \tag{1}$$

where $\Delta(\phi, k) = \sqrt{(1-k^2\sin^2\phi)}$,

was first remarked by Euler* (1766), although it was not until several years later that Landen† (1775) discovered in geometrical form the transformation which forms the basis of existing methods of numerical calculation of the elliptic integrals.

A method of successive transformations for the ultimate reduction of the algebraic forms of the elliptic integrals to elementary forms was published by Lagrange‡ in 1784–5. This memoir contains an exposition of the scales of arithmetico-geometrical means and discusses their use in calculating the elliptic integrals of the first and second kinds in a manner practically identical with that formulated about the same time by Legendre.*

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* Euler, 'Novi Comm. Acad. Sc. Petrop.,' vol. 10 (1766).

† Landen, 'Phil. Trans. Roy. Soc.,' vol. 65, p. 283 (1775); also, 'Mathematical Memoirs,' London, 1780.

‡ Lagrange, "Sur une nouvelle méthode de calcul intégral pour les differentielles affectées d'un radical carré sous lequel la variable ne passe pas le quatrième degré," 'Mém. de l'Acad. Roy. des Sc. de Turin,' vol. 2 (1784–5); 'Œuvres,' vol. 2, pp. 253–312 (Gauthier-Villars, Paris, 1868).

This method, discovered independently by Gauss and called by him the "algorithm of the arithmetico-geometrical mean," originated in connection with the evaluation of complete elliptic integrals arising from a problem in attractions occurring in the planetary theory.† In a note he mentions that the results were obtained by him as part of a more comprehensive theory independently of the results of Lagrange and Legendre, with which they are closely connected.

Much use was made of the scales of arithmetico-geometrical means in the numerical calculation of elliptical integrals by Jacobi,‡ who, however, adapted his formulæ to computation by logarithms. The use of modern calculating machines has greatly extended the scope of direct numerical calculation of elliptic integrals by the methods just referred to. In many practical applications it is advantageous to have available a direct method of computation independent of auxiliary tables, the use of which often requires tedious and somewhat uncertain interpolations when seven or more significant figures are desired. Many formulæ involving elliptic functions arising in mathematical physics, when expressed directly in terms of quantities arising from the arithmetico-geometrical scales, give rise to extremely convenient expressions for direct numerical computation when a calculating machine is available.

SECTION 2.—*The Scale of Arithmetico-geometrical Means.*

In forming the scale of arithmetico-geometrical means, we start with the positive numbers, $a_0 = 1$, $b_0 = k'$, $c_0 = k$, k being the modulus of formula (1), and k' the complementary modulus, $k'^2 = 1-k^2$. We calculate successively

$$\left.\begin{array}{lll} a_0 = 1, & b_0 = k', & c_0 = k \\ a_1 = \tfrac{1}{2}(a_0+b_0) & b_1 = \sqrt{(a_0 b_0)}, & c_1 = \tfrac{1}{2}(a_0-b_0) \\ a_2 = \tfrac{1}{2}(a_1+b_1) & b_2 = \sqrt{(a_1 b_1)}, & c_2 = \tfrac{1}{2}(a_1-b_1) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_n = \tfrac{1}{2}(a_{n-1}+b_{n-1}), & b_n = \sqrt{(a_{n-1}\, b_{n-1})}, & c_n = \tfrac{1}{2}(a_{n-1}-b_{n-1}) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \end{array}\right\} . \qquad (2)$$

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* Legendre, "Mémoire sur les intégrations par arcs d'ellipse,"—"Second mémoire, etc.," 'Mém. de l'Acad. des Sciences de Paris, année 1786,' pp. 616–643 (Paris, 1788); also pp. 644–683; 'Traité des Fonctions Elliptiques,' vol. 1, pp. 79 *et seq.* (Paris, 1825).

† Gauss, "Determinatio Attractionis, etc.," 'Comm. Gott. Soc. Reg. Scient.,' vol. 4, 1818; 'Werke,' vol. 3, pp. 331–355 (Göttingen, 1866). The note mentioned above is dated February 9, 1818; 'Werke,' vol. 3, pp. 357–360.

‡ Jacobi, 'Fundamenta Nova,' §§ 38 and 52 (1829); 'Ges. Werke,' vol. 1, pp. 154 and 203 (Berlin, 1881); also, "Numerische Berechnung der Elliptischen Functionen," 'Crelle,' vol. 26, pp. 93–114; 'Ges. Werke,' vol. 1, pp. 345–368 (1881).

The following relations are easily verified—

$$\left.\begin{array}{ll} c_n^2 = a_n^2 - b_n^2, & a_n c_n = \tfrac{1}{4} c_{n-1}^2 \\ a_{n+1} + c_{n+1} = a_n, & a_{n+1} - c_{n+1} = b_n \end{array}\right\}. \tag{3}$$

The array of numbers (2) will be briefly referred to as the A.G.M. scale ($a_0 = 1$, $b_0 = k'$). The successive a's and b's converge with extraordinary rapidity to the same limit, which Gauss denoted by $M(a_0, b_0)$. It is easily seen that

$$M(\epsilon a_0, \epsilon b_0) = \epsilon M(a_0, b_0). \tag{4}$$

The limit $M(a_0, b_0)$ will be denoted by a_n, as long as by so doing no ambiguity is involved.

The following expressions for the complete elliptic integrals in terms of the A.G.M. scale ($a_0 = 1$, $b_0 = k'$) are well known.

$$K = \tfrac{1}{2}\pi / a_n, \tag{5}$$

$$(K - E)/K = \tfrac{1}{2}(c_0^2 + 2c_1^2 + 4c_2^2 + \ldots + 2^n c_n^2 + \ldots). \tag{6}$$

It does not appear to have been remarked hitherto that $2^{-n}\log(a_n/c_n)$ also converges rapidly to a definite limit which enables us to compute the complementary integrals K′, E′, and the nome q given by $\log q = -\pi K/K'$.

It may be proved* that

$$\tfrac{1}{2}\pi\, a_n/a_n' = \tfrac{1}{2}\pi\, K'/K = 2^{-n}\log(a_n/c_n). \tag{7}$$

More convenient for calculation is the following formula, derived from (7) and the relations (3)

$$\log(1/q) = \pi K'/K = \log(4a_1/c_1) - \sum_{1}^{\infty} (\tfrac{1}{2})^{n-1} \log(a_n/a_{n+1}). \tag{8}$$

Making use of Legendre's relation,

$$E/K + E'/K' - 1 = \tfrac{1}{2}\pi/(KK'), \tag{9}$$

we easily derive the series

$$K' - E' = \tfrac{1}{2} a_n \left(\log\frac{4a_1}{c_1} - \log\frac{a_1}{a_2} - \tfrac{1}{2}\log\frac{a_2}{a_3} - \ldots\right)$$
$$(1 - \tfrac{1}{2}c_0^2 - c_1^2 - 2c_2^2 - 4c_3^2 - \ldots) - a_n. \tag{10}$$

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[*] The proof will be given in a memoir on the numerical calculation of elliptic functions which the writer has in preparation. The writer has been unable to find this formula explicitly stated by Gauss or by any subsequent writer on elliptic functions. It furnishes the key, however, to many remarkable formulæ found among Gauss's unpublished papers, stated without proofs ('Werke,' vol. 3 (1866)); Nachlass, 'Arithmetisch-Geometrisches Mittel,' pp. 361–402; also, 'Zur Theorie der Neuen Transscendenten,' pp. 446 *et seq.*). The series (8) is, however, given without proof by Gauss (*loc. cit.*, p. 377), who may have been in possession of (7), but for some reason disposed to withhold the publication of this key-formula.

SECTION 3.—*Application to Mutual Inductance Formulæ for Coaxial Circles.*

Let A be the radius of the larger circle, a that of the smaller, and d the distance between their planes. We also denote by r_1 and r_2 the greatest and least distances respectively of a point on one circle to the circumference of the other. Maxwell* derived two well-known formulæ for M, the coefficient of mutual induction of the two circles;

$$\left.\begin{array}{lll} & \mathrm{M} = 8\pi(\mathrm{K}/k)\sqrt{(\mathrm{A}a)}\,[1-\mathrm{E}/\mathrm{K}-\tfrac{1}{2}k^2] & \\ \text{where} & k^2 = 4\mathrm{A}a/\{(a+\mathrm{A})^2+d^2\}, & k' = r_2/r_1 \\ \text{and} & r_1^2 = (\mathrm{A}+a)^2+d^2, & r_2^2 = (\mathrm{A}-a)^2+d^2 \end{array}\right\}. \tag{11}$$

Landen's transformation applied to (11) gives rise to Maxwell's second formula

$$\mathrm{M} = 8\pi\sqrt{(\mathrm{A}a/k_1)}\,(\mathrm{K}_1-\mathrm{E}_1), \tag{12}$$

where K_1 and E_1 are the complete elliptic integrals to modulus

$$k_1 = (r_1-r_2)/(r_1+r_2). \tag{13}$$

If we compute the A.G.M. scale ($a_0 = r_1$, $b_0 = r_2$), it is readily proved from (4), (5), and (6) that both (11) and (12) give rise to

$$\mathrm{M} = \{2\pi^2/a_n\}\,[c_1^2+2c_2^2+4c_3^2+\ldots+2^{n-1}c_n^2+\ldots], \tag{14}$$

a most convenient and elegant formula for the direct computation of M.

This series is always extremely convergent, since from (7) the successive values of c tend to converge like the series

$$x \quad x^2 \quad x^4 \quad x^8 \quad x^{16}\ldots x^{2^n}\ldots \quad \text{where } x<1.$$

When the circles are *very close together*, k' is small and it is more convenient to use the *complementary* A.G.M. scale, which is then extremely convergent. The first few terms are as follows:—

$$\left.\begin{array}{cccc} n & a_n' & b_n' & c_n' \\ 0 & r_1+r_2 & r_1-r_2 & 2\sqrt{(r_1r_2)} \\ 1 & r_1 & 2\sqrt{(\mathrm{A}a)} & r_2 \\ 2 & \tfrac{1}{2}r_1+\sqrt{(\mathrm{A}a)} & (4\mathrm{A}ar_1^2)^{\frac{1}{4}} & \tfrac{1}{2}r_1-\sqrt{(\mathrm{A}a)} \end{array}\right\}. \tag{15}$$

Making use of (10) and (12), we easily obtain the following formula for M in terms of the complementary A.G.M. scale (denoted by accented letters):—

$$\mathrm{M} = 2\pi a_n'\left[\frac{(r_1^2-2c_2'^2-4c_3'^2-\ldots)}{a_n'^2}\left(\log\frac{4a_2'}{r_2}-\tfrac{1}{2}\log\frac{a_2'}{a_3'}-\tfrac{1}{4}\log\frac{a_3'}{a_4'}-\ldots\right)-2\right]. \tag{16}$$

Logarithms to the base 10 may be used in computing the second series, which may finally be multiplied by the factor $\mu = \log_e 10 = 2{\cdot}30258509$ to convert to logarithms to base e.

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* Maxwell, 'Electricity and Magnetism,' 3rd ed., sec. 701 (1904).

For accuracy the first term of the above formula should exceed twice the second term. This will be the case if $r_1/r_2 > 13$, about. Hence formula (16) is suitable for computation when the circles are so close together that $r_2/r_1 < 0{\cdot}1$. It may, without serious loss of accuracy, be extended to cases where $r_2/r_1 < 0{\cdot}2$, in which case seven-figure accuracy only requires the complementary A.G.M. scale (15) to be computed as far as $n = 2$.

The formula (14) is, however, so much simpler than (16) that it may be used even when the circles are very close together. It has the advantage of depending only on figures derived by direct computation on a calculating machine, and does not depend on the use of logarithmic tables.

By dividing up coils of finite cross-section into a number of coils of sufficiently small cross-section, we may make use of a formula of quadratures for computing the mutual induction, depending on the calculation of the coefficients for the elementary coils taken in pairs. For this purpose the simple formula (14) is specially applicable, while (16) may be useful in some cases. Extremely convergent formulæ for the numerical computation of the induction coefficients of coaxial single-layer coils may be obtained by the use of A.G.M. scales and trigonometrical recurrence-formulæ. These depend on a number of new formulæ for the computation of the incomplete elliptic integrals of the first, second, and third kinds, which it is hoped to deal with in a future paper.

SECTION 4.—*Numerical Illustrations.*

To illustrate the rapid convergence of the A.G.M. scales, Gauss* works out an example, starting with $a_0 = 1$, $b_0 = 0{\cdot}2$, and finds that a_5 and b_5 are represented by the same number to fifteen significant figures.

The usual arithmetical process of extracting square roots is easily adapted to calculating machines of modern construction.† In carrying out the computations, the work is considerably shortened by keeping in mind the rule that if the first p out of the n digits of the number required in the square root have been obtained by the usual process, the next $(p-1)$ digits can be obtained by division only, with a possible error of 1 in the last digit.‡

The following numerical examples, taken from the collection of formulæ published by the Washington Bureau of Standards,* illustrate the advantages of formulæ (14) and (16) in numerical computation.

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* Gauss, 'Werke,' vol. 3, p. 363 (1866).

† A description of modern calculating machines is given by Horsburgh, E. M., 'Modern Instruments and Methods of Calculation' (London, G. Bell and Sons, 1914); also by d'Ocagne, 'Le Calcul Simplifié' (Gauthier-Villars, Paris, 1905).

‡ Chrystal's 'Algebra,' Part I, 5th ed., p. 210 (1904)

Example 1. 'Bulletin 169,' p. 32 (1916). *Equal circles at a distance.*

$A = a = 25$ cm., $d = 50$ cm., $r_1 = 50\sqrt{2}$ cm., $r_2 = 50$ cm.

n.	a_n.	b_n.	c_n.	c^2_n.
0	70·71067812	50·00000000	—	—
1	60·35533909	59·46035575	10·35533906	$107{\cdot}23304776_1$
2	59·90784740	59·90617607	0·44749167	$0{\cdot}20024879_5$
3	59·90701174	59·90701173	0·00083567	$0{\cdot}00000069_8$

To an order of accuracy of ten significant figures, we need take only three terms of (14), which then gives, with $2\pi^2 = 19{\cdot}7392088022$,

$$M = \frac{2\pi^2}{a_4}(c_1^2+2c_2^2+4c_3^2) = \frac{2\pi^2\times 107{\cdot}63354814_3}{59{\cdot}90701173_5} = 35{\cdot}46498187_8 \text{ cm.}$$

It is readily seen that little additional labour is involved in carrying out the computations to ten significant figures. Values quoted in the above-mentioned 'Bulletin 169,' computed by three different formulæ expressed in q-series, derived by Nagaoka, give for M the three values

35·464975 cm., 35·464981 cm., 35·46481 cm.

In carrying out the computation, there is no need to write down more figures than are set down above, all the processes being carried out on the calculating machine.

Example 2. 'Bulletin 169,' pp. 21, 22, 28 (1916). *Equal circles close together.*

$A = a = 25$ cm., $d = 1$ cm., $r_1 = 50{\cdot}010000$. $r_2 = 1{\cdot}00000000$.

From (15) we write down the complementary A.G.M. scale—

n.	a_n'.	b_n'.	c'_n.
1	50·010000	50·000000	1·000000
2	50·005000	50·005000	0·005000

Formula (16) then gives to eight significant figures.

$$M = 2\pi\left[\left(\frac{r_1^2-2c_2'^2}{a_3'}\right)\log\frac{4a_2'}{r_2} - 2a_3'\right]$$

$$= 2\pi\left[\frac{2501{\cdot}0000}{50{\cdot}005000}\times\log(200{\cdot}02000) - 100{\cdot}01000\right],$$

or $$M = \pi\left[\frac{1000{\cdot}4000}{10{\cdot}001000}\times 5{\cdot}2984174 - 200{\cdot}0200\right] = 1036{\cdot}6648_6 \text{ cm.}$$

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* Bulletin 169, 'Formulas and Tables for the Calculation of Mutual and Self-Inductance,' by Rosa and Grover, 3rd ed., 1916.

Values derived from Legendre's tables and two series formulæ are given as

1036·6664 cm., 1036·6652 cm., 1036·663 cm.

A comparison of the numerical work involved emphasises in marked manner the advantage of the new formula (16).

It is interesting to check this result by formula (14), although the present example is not favourable to its use.

The computation proceeds as follows:—

n.	a_n.	b_n.	c_n.	c_n^2.
0	50·01000$_0$	1·00000$_0$	—	—
1	25·50500$_0$	7·07177$_4$	24·50500	600·4950
2	16·28838$_7$	13·43000$_0$	9·21661	84·9459
3	14·85919$_3$	14·79030	1·42919	2·0426
4	14·82475	14·82471	0·03445	0·0012
5	14·82473	14·82473	0·00002	0·0000

Even in this unfavourable case, four terms of (14) are more than sufficient to give an accuracy to seven significant figures, and

$$\mathrm{M} = \frac{2\pi^2}{a_5}(c_1^2+2c_2^2+4c_3^2+8c_4^2) = \frac{2\pi^2 \times 778{\cdot}5666}{14{\cdot}82473} = 1036{\cdot}666 \text{ cm.}$$

The labour of computation by the above formula is still less than that involved in the use of Legendre's Tables or of any of the series formulæ.

SECTION 5.—*Summary and Conclusions.*

The use of the modern calculating machine for the extraction of square roots makes the *scales of arithmetico-geometrical means* extremely convenient for the direct and rapid computation of the complete elliptic integrals.

This procedure is applied to Maxwell's formulæ for the coefficient of mutual induction of coaxial circles, with the result that two new and elegant formulæ are obtained for numerical computation.

Numerical examples, taken from the well-known collection of formulæ and tables for the calculation of self and mutual induction, issued by the Bureau of Standards, illustrate the advantage of the new formulæ in economy of labour.